DAMPING OF DRIFT WAVES

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The damping of a monochromatic wave propagating at an angle to the magnetic field in an inhomogeneous plasma is studied. The nonlinear equations for resonance particles are solved in the drift approximation. The nonlinear damping decrement is calculated.

We consider the damping of a drift wave propagating in a plasma with density and temperature gradients which are in the same direction ($d\ln T/d\ln n > 0$). We take the ion temperature to be zero. We restrict ourselves to the case of quasi-neutral potential oscillations

$$\omega / k_z \ll v_{Te} < c_A, c_A = (H_0^2 / 4\pi n_0 m)^{1/2}. \tag{1}$$

Here c_A is the Alfven velocity and k_z is the wave-vector component in the direction of a constant magnetic field.

We define the nonlinear damping decrement γ as the ratio of the work per unit time A performed by the electric field of the wave on the resonance particles in the localization region, to the energy of the wave. As in [1-3], we solve the nonlinear equations of motion for the resonance particles on the assumption that the wave amplitude is independent of time. We take the energy of the wave W to be equal to the energy of the wave at the initial moment of time W_0

$$\gamma = \frac{A(E_0)}{2W_0} \qquad (W_0 = W(E_0)). \tag{2}$$

For frequencies close to the drift frequency

$$W_0 = \int \frac{1}{4} \frac{e^2 \varphi_0^2(x)}{kT(x)} n_0(x)\, dx.$$

Here $\varphi_0(x)$ is the potential amplitude at the initial moment of time. In the quasiclassical approximation $\varphi_0(x)$ has the following form [4]:

$$\varphi_0(x) = CQ^{-1/4} \exp\left(i \int k_x dx\right), \qquad k_x = k_x' + ik_x'' = \sqrt{Q(x, \omega)}$$

$$Q(x, \omega) = -k_y^2 + \frac{k_z^2 \omega_{Hi}^2}{\omega^2} + \frac{\omega_{Hi} M}{v_{Te}^2 m} - \frac{k_y \omega_{Hi}}{\omega} \frac{d \ln n_0}{dx} + i \sqrt{\frac{\pi}{2}} \frac{M \omega_{Hi}^2}{m k_z v_{Te}^3} \left(-\omega + \frac{k_y v_{Te}^2}{\omega_{He}} \frac{d}{dx} \ln \frac{n_0(x)}{\sqrt{T(x)}}\right). \tag{3}$$

In order to determine the work performed by the wave-field on the resonance particles we have to solve the equations of motion of the particles. Let us write these equations in a system of coordinates which is moving along the axis (the direction of the constant magnetic field) with a velocity ω/k_z:

$$\frac{dv_x}{dt} = \frac{e}{m} \frac{d\varphi_0(x)}{dx} \cos(k_y y + k_z z') - \omega_{He} v_y \qquad \left(z' = z - \frac{\omega}{k_z} t\right) \tag{4}$$

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$$\frac{dv_y}{dt} = -\frac{e}{m}\varphi_0(x)\,k_y \sin(k_y y + k_z z') + \omega_{He} v_x$$

$$\frac{du}{dt} = -\frac{e}{m}\varphi_0(x)\,k_z \sin(k_y y + k_z z'), \quad u = \frac{dz'}{dt} = v_z - \frac{\omega}{k_z}\,.$$

We restrict ourselves to the drift approximation (neglecting the particle inertia for motion across the magnetic field). We then obtain the simpler system of equations in place of (4):

$$\frac{dx}{dt} = \frac{ck_y}{H_0}\varphi_0 \sin(k_y y + k_z z'), \quad \frac{dy}{dt} = \frac{c}{H_0}\frac{d\varphi_0}{dx}\cos(k_y y + k_z z')$$
$$\frac{du}{dt} = \frac{e}{m}\varphi_0 k_z \sin(k_y y + k_z z'), \quad \frac{dz'}{dt} = u\,. \tag{5}$$

System (5) has two solutions. Multiplying the third equation by $(k_y/k_z\omega_{He})$ and subtracting it from the first we obtain the first solution

$$x + \frac{uk_y}{\omega_{He}k_z} = q\,. \tag{6}$$

Introducing the new variable

$$\psi = k_y y + k_z z'$$

we have

$$\frac{d\psi}{dt} = k_z u + \frac{ck_y}{H_0}\frac{d\varphi_0}{dx}\cos\psi\,. \tag{7}$$

Using Eq. (7) and the third equation of (5) we find the second solution for the system

$$\tfrac{1}{2}\, mu^2 - e\varphi_0(x)\cos\psi = E' = \text{const.} \tag{8}$$

It follows from (6) and (8) that the displacement of the resonance particles $(u \sim \sqrt{2e\varphi_0/m})$ on the x axis is of order

$$\Delta x \sim (k_y/k_z\omega_{He})\sqrt{2e\varphi_0/m}\,.$$

If we assume that

$$(k_x' k_y/k_z\omega_{He})\sqrt{(2e\varphi_0/m)} \ll 1 \tag{9}$$

then Eqs. (6)-(8) may be reduced to quadratures.

Expanding $\varphi_0(x)$, $d\varphi_0/dx$ in a power series of $\Delta x = (uk_y/\omega_{He}k_z)$ and confining ourselves to first-order terms in (6)-(8) we have

$$\frac{m}{2k_z^2}\left(\frac{d\psi}{dt}\right)^2 - e\varphi_0(q)\cos\psi = E'. \tag{10}$$

An equation of this type also occurs in the problem of the damping of plasma oscillations [1,2].

As in [2] Eq. (10) can be reduced to a form which is more convenient for integration by introduction of the new variable $\xi = \frac{1}{2}\psi$:

$$\dot{\xi}^2 = \gamma^{-2}\tau^{-2}(1 - \chi^2 \sin^2\xi)\,. \tag{11}$$

Here

$$\chi^2 = \frac{2e\varphi_0(q)}{E' + e\varphi_0(q)}, \qquad \tau = \left(\frac{m}{e\varphi_0}\right)^{1/2} k_z^{-1}\,.$$

For the case in which $\chi^2 < 1$ the solution can be written down immediately as

$$F(\chi, \xi_0) = F(\chi, \xi) - (t/\chi\tau)\,. \tag{12}$$

For $\chi^2 > 1$ the solution can also be written down with the help of elliptic integrals if we use the new variable ζ instead of ξ,

$$\chi \sin \xi = \sin \zeta, \quad \zeta^{\cdot 2} = \tau^{-2} (1 - \chi^2 \sin^2 \zeta), \; F(\chi^{-1}, \zeta_0) = F(\chi^{-1}, \zeta) - t/\tau . \tag{13}$$

The work performed per unit time by the electric field on the resonance particles, or the change per unit time of kinetic energy of the resonance particles is*

$$A(E_0) = \frac{dT}{dt} = \int dx \int_0^{2\pi} \frac{d\psi}{2\pi} \int_{-\infty}^{\infty} du \, \frac{m}{2} \left(u + \frac{\omega}{k_z}\right)^2 \frac{\partial f}{\partial t} . \tag{14}$$

Let us write the distribution function at any moment of time in the form

$$\begin{gathered} f(x, \psi, u, t) = f_0 \, [x_0(x, \psi, u, t), \psi_0(x, \psi, u, t), u_0(x, \psi, u, t), 0] \\ f_0 = n_0(x_0) \left[\frac{m}{2\pi k T(x_0)}\right]^{1/2} \exp\left[-\frac{m}{2kT(x_0)}\left(u + \frac{\omega}{k_z}\right)^2\right] . \end{gathered} \tag{15}$$

Here x_0, ψ_0, u_0 are the coordinates and velocity for $t = 0$.

Using the equations of motion we find

$$\begin{gathered} \frac{\partial f}{\partial t} = -\frac{e}{m} \varphi_0(x_0) k_z \sin \psi_0 \left[\frac{\partial f_0}{\partial u_0} + \frac{k_y}{\omega_{He} k_z} \frac{\partial f}{\partial x_0}\right] \\ \psi_0 = k_y y_0 + k_z z_0 . \end{gathered} \tag{16}$$

Differentiating (16) with respect to u_0 and x_0 and taking into account the fact that $u_0 \sim \sqrt{2e\varphi_0/m}$ in the resonance region, we obtain the following expression for the nonlinear damping decrement:

$$\begin{gathered} \gamma = \frac{1}{2W_0} \int dq \int_0^{2\pi} \frac{d\psi}{2\pi} \int_{-\infty}^{\infty} \left(u + \frac{\omega}{k_z}\right)^2 f_0\left(x_0, \frac{\omega}{k_z}\right) \frac{e\varphi(x_0)}{2} S \sin \psi_0 \\ S = \frac{m\omega}{kT(x_0) k_z} - \frac{k_y}{\omega_{He} k_z n_0(x_0)} \frac{dn_0}{dx} \left[1 - \frac{1}{2} \frac{d \ln T(x_0)}{d \ln n_0(x_0)}\right] \\ \varphi(x_0) = \varphi(q) - \frac{2k_y}{\omega_{He} \chi \tau k_z} \left[1 - \chi^2 \mathrm{cn}^2 \left(F(\chi, \xi) - \frac{t}{\chi\tau}, \chi\right)\right]^{1/2} \frac{d\varphi_0}{dq} \\ \sin \psi_0 = 2 \sin [F(\chi, \xi) - t/\chi\tau, \chi] \, \mathrm{cn} \, [F(\chi, \xi) - t/\chi\tau, \chi] \quad (\chi^2 < 1) \\ \varphi(x_0) = \varphi_0(q) - \frac{2k_y}{\omega_{He}\chi\tau k_z} \mathrm{cn}\left[F(\chi^{-1}, \zeta) - \frac{t}{\tau}, \chi^{-1}\right] \frac{d\varphi_0}{dq} \\ \sin \psi_0 = 2\chi^{-1} \, \mathrm{sn} \, [F(\chi^{-1}, \zeta) - t/\tau, \chi^{-1}] \, \mathrm{dn} \, [F(\chi^{-1}, \zeta) - t/\tau, \chi^{-1}] \, (\chi^2 > 1) . \end{gathered} \tag{17}$$

If we neglect the displacement of the particles Δx compared with $(k_x^{\prime})^{-1}$ and take the values of density and temperature at the midpoint of localization $x = 0$, then (17) can be reduced to a very simple form:

$$\gamma = \frac{\sqrt{\pi}\,\omega^{*2} \eta(0)}{2k_z v_{Te}(0)} P(t), \quad \omega^* = \frac{kT(0)}{M\omega_{Hi}} \frac{1}{n_0(0)} \frac{dn_0(0)}{dx}, \quad \eta(0) = \frac{d \ln T(0)}{d \ln n_0(0)} \tag{18}$$

$$\begin{gathered} p(t) = \sum_{n=0}^{\infty} \frac{64}{\pi} \int_0^1 d\chi \left\{\frac{2n\pi^2 \sin(\pi n t / \chi\tau F)}{\chi^5 k^2 (1 + q^{2n})(1 + q^{-2n})} + \frac{(2n+1)\pi^2 \chi \sin[(2n+1)\pi t / 2\chi\tau F]}{k^2 (1 + q^{2n+1})(1 + q^{-(2n+1)})}\right\}, \\ q = \exp \frac{\pi k'}{k}, \quad k = F(\chi, {}^1/{}_2\pi) . \end{gathered} \tag{19}$$

It was shown in [2] that $p(t) \to 1$ as $t \to 0$.

Using Eqs. (17) we can obtain an expression for the decrement which is more exact than (18); in so doing it is sufficient to take terms of the order $k_x^{\prime} \Delta x$ into account.

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*In what follows we use the method developed in [2] for calculating the decrement.

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